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# On the Expansion of the Single Eigenvalue Probability Density Function

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The probability density function of the single eigenvalue is expanded in terms of the reciprocal of the dimension of the matrix using Bessel functons. It is shown that for the new matrix ensembles this expansion gives Wigner's semicircle centered at the mean value of the matrix elements plus terms of the order of $N^{-1}$, where $N$ is the dimension of the matrix.

**KEY WORDS:** Matrix ensembles; probability distributions; eigenvalue distribution.

## 1. INTRODUCTION

Matrix ensembles have been successfully employed in the past to study the statistical properties[1] of compound-nucleus level widths and level densities. Since compound-nucleus levels lie much higher in energy, it was reasonable to assume that the off-diagonal elements of the Hamiltonian will have almost equal probability of being positive and negative. Thus in these ensembles the mean value of the off-diagonal elements was always taken to be zero. Later, [2] in an attempt to understand the average properties of complex nuclei using such ensembles, it was found that for a satisfactory description of these complex nuclei one has to modify these ensembles by assuming that the off-diagonal elements have nonzero mean. The nonzero-mean matrix ensembles are also finding applications in other areas[3] of many-body physics. Since in all these studies the dimension $N$ of the matrix is taken to be large, one is interested in expanding the various distributions in terms of the inverse of the dimension of the matrix. An

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important quantity which is also needed in the study of average cross sections is the probability density function of the single eigenvalue for large $N$. The purpose of the present work is to look into the problem of the expansion of various probability density functions in terms of $N^{-1}$.

To bring out the essential points of the present way of looking at this problem, we shall first derive the distribution of a component of an $N$-dimensional unit vector in Section 2 for large $N$. In Section 3 we shall consider the probability density function when the off-diagonal elements have zero mean. The case in which the mean value of the matrix elements is nonzero will be discussed in Section 4.

## 2. DISTRIBUTION OF A COMPONENT OF AN $N$-DIMENSIONAL VECTOR

Let us consider an $N$-dimensional unit vector having the components $x_1, x_2, \ldots, x_N$. The probability density function of a single component, say $x_1$, can then be written as

$$P(x) = K\int \delta(x - x_1)\delta\left(\sum_{i=1}^{N} x_i^2 - 1\right)\prod_{i=1}^{N} dx_i \tag{1}$$

where $K$ is the normalization constant. In all the subsequent equations $K$ will stand for the appropriate normalization constant. Before we derive the expansion of $P(x)$ in terms of $N^{-1}$, we would like to remark that exact distribution $P(x)$ can be easily obtained using $N$-dimensional polar coordinates.[4] Such exact distributions cannot always be derived for other quantities, like the single eigenvalue distribution when the mean value of the off-diagonal elements is nonzero. It is for this reason that we have chosen this problem so that we can first check the validity of the approximation by comparing it with the exact results and then use it later for other, more complicated distributions.

Writing the Fourier transform of the first delta function in expression (1), we get

$$P(x) = K\int \exp[-ik(x - x_1)]\delta\left(\sum x_i^2 - 1\right)\prod_{i=1}^{N} dx_i\, dk \tag{2}$$

where $K$ now denotes the new constant.

Expanding $\exp(ikx_1)$ and keeping the first three nonvanishing terms, we get

$$P(x) = K\int \exp(-ikx)\left\{1 - \frac{1}{2!}k^2x_1^2 + \frac{1}{4!}k^4x_1^4 \cdots\right\}\delta\left(\sum x_i^2 - 1\right)\prod_i dx_i\, dk \tag{3}$$

Carrying out the integrations over $x_i$ gives

$$P(x) = K\int \exp(-ikx)\left[1 - \frac{1}{2N}k^2 + \frac{1}{8N(N+2)}k^4 \cdots \right] dk \tag{4}$$

Introducing the new random variable $t = N^{1/2}x$ and making a slight change in the integral in expression (4), we arrive at the following probability density function of the variable $t$ for large values of $N$:

$$P(t) = K\int dk \exp(-ikt)\left[1 - \tfrac{1}{2}k^2 + \tfrac{1}{8}k^4(1 - 2/N) \cdots \right] \tag{5}$$

We see from expression (5) that the series in the square brackets is a sum of a dominant part represented by $(1 - \frac{1}{2}k^2 + \frac{1}{8}k^4 \cdots)$ and small terms of the order of $1/N$ and therefore provides the desired expansion of the variable $t$ for large values of $N$. Rewriting expression (5) in the form

$$P(t) = K\int dk \exp\left(-ikt - \tfrac{1}{2}k^2\right)\left[1 - (1/4N)k^4 \cdots \right] \tag{6}$$

and carrying out the integration over $k$, we get the following expression for $P(t)$ for large $N$:

$$P(t) = K\exp\left(-\tfrac{1}{2}t^2\right)\left[1 + (3/2N)t^2 - (1/4N)t^4 \cdots \right] \tag{7}$$

Before we go to the next section let us note the following: (i) By expanding the exact distribution

$$P_{\text{ex}}(t) = K\left(1 - t^2/N\right)^{(N-3)/2} \tag{8}$$

we find that our procedure has given the correct expansion in terms of $N^{-1}$. (ii) The exact range of the variable $t$ is $-N^{1/2} < t < N^{1/2}$, while for the asymptotic form (7) it is $-\infty < t < \infty$. (iii) It is obvious that expression (7) reproduces the second and fourth moments of $t$ up to order $N^{-1}$ correctly, as it should.

## 3. MATRIX ENSEMBLES HAVING ZERO MEAN

Let us consider a real-symmetric matrix ensemble in which each matrix element is distributed according to[3]

$$P(H_{ij}) = \left(2\pi J^2/N\right)^{-1/2} \exp\left(-NH_{ij}^2/2J^2\right) \tag{9}$$

This is a slightly different distribution from the one used in earlier studies,[1] which gives rise to a Wishart distribution for the eigenvalues. For the Wishart distribution Wigner had derived the dominant part of the single eigenvalue distribution using methods of statistical mechanics and the method of moments.[5] An exact expression for the probability density function was later derived by Mehta and Gaudin[6] by carrying out the exact integrations. Such exact derivation will be extremely difficult when each element $H_{ij}$ has a finite mean, since even the joint distribution of the eigenvalues can only be written as a Wishart distribution multiplied by a factor which is an integral over a unit vector[7]; however, the method of Section 2 provides a fairly straightforward way to find the expansion of the probability density function even for this case.

The probability density function $P(E)$ can be written as

$$P(E) = K\int \mathrm{Tr}[\delta(E-H)]P(H)\,dH \tag{10}$$

where Tr denotes the trace and $P(H)$ is the product of probabilities of $\frac{1}{2}N(N+1)$ elements given by expression (9) and $K$ again denotes the appropriate normalization constant. Since all the matrix elements of $H$ are treated alike, expression (10) can be rewritten as

$$P(E) = K\int [\delta(E-H)]_{11}P(H)\,dH \tag{11}$$

where $[\cdots]_{11}$ denotes the matrix element of $\delta(E-H)$ lying in the first row and first column. As in Section 2, we write the Fourier transform of the $\delta$-function and expand the exponential $[\exp ikH]_{11}$ to get

$$P(E) = K\int dk\exp(-ikE)\Big(1-\frac{1}{2!}k^2[H^2]_{11}+\frac{1}{4!}k^4[H^4]_{11}\Big)P(H)\,dH \tag{12}$$

Carrying out the integrations over $dH$ and retaining terms of the order of $N^{-1}$, we find

$$P(E) = K\int dk\exp(-ikE)\Big[\Big(1-\frac{1}{2}k^2J^2+\frac{1}{12}k^4J^4\cdots\Big)-\frac{1}{12N}k^4J^4\cdots\Big] \tag{13}$$

As in Section 2, the probability density function of the single eigenvalue $E$ is a sum of the dominant part arising from the terms $(1-\frac{1}{2}k^2J^2\cdots)$ and the part which goes like $N^{-1}$ for large $N$. Using Bessel functions[8] and making a slight change in the integral in expression (13), we can write

$$P(E) = K\int_{-\infty}^{\infty} dt\,t^{-1}\exp\Big(-\frac{itE}{2J}\Big)\Big[J_1(t)-\frac{10}{N}J_5(t)\cdots\Big] \tag{14}$$

The integral over $t$ can be written[8] in terms of Chebyshev polynomials, which finally gives the following expression for $P(E)$:

$$P(E) = K[4J^2-E^2]^{1/2}\Big[1+\frac{1}{N}\Big(\frac{6E^2}{J^2}-\frac{2E^4}{J^4}\Big)\cdots\Big] \tag{15}$$

$|E|<2J$ and $P(E)$ being zero outside this interval. The first part of the probability density function is just Wigner's semicircle,[5] as it should be, since it is the large number of off-diagonal elements that determines this part. The second part of expression (15) goes as $N^{-1}$ and it can be easily checked that it gives correct second- and fourth-order moments of $E$ up to $N^{-1}$.

## 4. MATRIX ENSEMBLES HAVING NONZERO MEAN

We would next like to consider matrix ensembles in which each $H_{ij}$ has a nonzero mean.[3] Expression (9) is now replaced by

$$P(H_{ij}) = \left(\frac{2\pi J^2}{N}\right)^{-1/2} \exp\left[-\frac{N}{2J^2}\left(H_{ij} - \frac{M_0}{N}\right)^2\right] \tag{16}$$

The probability density function is again given by expression (10) or (11) with $P(H)$ being the product of probabilities of $\frac{1}{2}N(N+1)$ elements given by expression (16). Since the diagonal elements of $H$ have now a mean value $M_0/N$, we rewrite expression (11) as

$$P(E) = K\int \delta\left[\left(E - \frac{M_0}{N}\right) - \left(H - \frac{M_0}{N}\right)\right]_{11} P(H)\,dH \tag{17}$$

Again writing the Fourier transform of the $\delta$-function and expanding $[\exp ik(H - M_0/N)]_{11}$, we get

$$P(E) = K\int dk \exp\left[-ik\left(E - \frac{M_0}{N}\right)\right]\left\{1 + ik\left[H - \frac{M_0}{N}\right]_{11} - \frac{k^2}{2!}\left[\left(H - \frac{M_0}{N}\right)^2\right]_{11} - i\frac{k^3}{3!}\left[\left(H - \frac{M_0}{N}\right)^3\right]_{11} + \frac{k^4}{4!}\left[\left(H - \frac{M_0}{N}\right)^4\right]_{11} + \cdots\right\} \tag{18}$$

The rest of the derivation is almost the same as described in the last section. The second term in the curly brackets of expression (18) contributes zero. The third and fifth terms give rise to the same dominant terms as in expression (13). The $N^{-1}$ terms which arise from the third, fourth, and fifth terms now involve the Bessel functions $J_3$, $J_4$, and $J_5$. The final expression which we obtain for $P(E)$ is given by

$$P(E) = K\left[4J^2 - \left(E - \frac{M_0}{N}\right)^2\right]^{1/2}\left[1 + \alpha\left(E - \frac{M_0}{N}\right) + \beta\left(E - \frac{M_0}{N}\right)^2 + \gamma\left(E - \frac{M_0}{N}\right)^3 + \delta\left(E - \frac{M_0}{N}\right)^4 + \cdots\right] \tag{19}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are given by

$$\alpha = -(2/NJ^4)(M_0^3 + 3M_0J^2) \tag{20a}$$

$$\beta = (1/NJ^6)(6J^4 - 2M_0^2J^2 - 3M_0^4) \tag{20b}$$

$$\gamma = (1/NJ^6)(M_0^3 + 3M_0J^2) \tag{20c}$$

$$\delta = -(1/NJ^8)(2J^4 - M_0^2J^2 - M_0^4) \tag{20d}$$

Therefore the probability density function when each element of $H_{ij}$ has a mean value $M_0/N$ is given by Wigner's semicircle centered at $M_0/N$ plus terms of the order $N^{-1}$ for large $N$.

## 5. CONCLUSIONS

The probability density function for various quantities is separated into a dominant part and a small part which goes as $N^{-1}$ using appropriate expansions in terms of Hermite polynomials with Gaussian weight factor and Bessel functions. Even though we have discussed the distribution of the single component of a unit vector and the single eigenvalue the method can also be applied to find other distributions, like the joint distribution of two components of an $N$-dimensional unit vector. Such distributions are needed in the study of correlations of the widths of the compound-nucleus levels.

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